Stable motions around triangular libration points in the real Earth–Moon system

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ABSTRACT
Stable motions around the triangular libration points (TLP) in the Earth–Moon system perturbed by Sun are described. Two special quasi-periodic orbits around each TLP (called dynamical substitutes in this paper) are given. They were previously described in the planar Sun–Earth–Moon system by Kamel and are numerically obtained in the spatial Sun–Earth–Moon system. Linearized motions around the dynamical substitute are described in a semi-analytical way. The size of the stable region around the substitute is described via a numerical approach, along with the resonance mechanisms that determine the boundary of the stable region. Possible temporary capture from the near Earth objects population is discussed. An observation algorithm to search permanent or temporarily captured objects in this region is given. Some potential applications of the TLPs of the Earth–Moon system in space missions are discussed.

Key words: minor planets, asteroids: general.

1 INTRODUCTION
Trojans are a special kind of small bodies in the Solar system and are located around the Lagrangian triangular libration points (TLPs) of the restricted three-body problem composed of the Sun and the planets. Till now, Trojans have been found in Sun–Jupiter, Sun–Earth, Sun–Mars, Sun–Neptune, and even Sun–Uranus system (Alexandersen et al. 2013).

For more than 50 years, people tried to answer this question: whether there are ‘Trojans’ around the TLPs in the Earth–Moon system. In 1961, the Poland astronomer Kordylewsky (1961) reported that he found some brighter regions (compared to the background) in the sky during his photographic observations of the area around the L5 point in the Earth–Moon system. Some observers confirmed his observations (Simpson 1967; Roach 1975; Winiarski 1989), while some others failed (Roosen 1968; Roosen & Wolf 1969). These suspicious objects are now generally referred to as the ‘Kordylewsky cloud’ in literature. There is still a debate about their existence. One opinion is that even if these objects ever existed, they are only temporary, i.e. the orbits are unstable and will eventually escape from this region after some time. After 1980s, little work has been published about these objects. In the 1990s, the Japanese explorer Hiten passed the proximity of these points and found no obvious accumulation of matter there (Igenbergs et al. 1991).

To the authors’ knowledge, all positive reports regarding the Kordylewsky cloud only lasted a few days, with no positive reports lasting one month or longer having been published. Besides, except a few of them (Freitas & Valdes 1980), all the observations focus on the area very close to the TLPs. Neglecting mutual forces between small bodies, previous studies (for example, see Díez, Jorba & Simó 1991; Hou & Liu 2010) indicate that motions close to the TLPs in the Earth–Moon system are actually unstable due to perturbations from the Sun. As a result, it is not possible for small bodies to occupy the region close to the TLPs.

Nevertheless, current studies in this paper indicate that there are stable orbits around but far away from the libration points. If natural objects really exist, these stable orbits may be good candidates to host them. As we will see, the amplitudes of these stable orbits are very large (on the order of the Earth–Moon distance). For an observer on the Earth, there are only a few days in one month that these orbits are located in the proximity of the libration points. As a result, focusing on the proximity of the TLPs is not always a good choice. The best way to search objects is to follow these stable orbits. In Section 5, an observation strategy will be given.

In the following, we first describe these stable orbits and depict their stable region. Then, we will describe the temporary capture phenomenon. After that, an observation strategy is proposed to search for permanent or temporarily captured objects. At the end of the
paper, some potential applications of the TLPs in the Earth–Moon system are discussed.

2 MODELS DESCRIPTION

In our work, two force models will be used. The first model is the Sun–Earth–Moon system, and the second model is the full force model which also incorporates the other seven planets in the Solar system. The first model is given by integrating the orbits of the Moon and the Sun with respect to the Earth under their mutual gravitations. Their initial conditions are given by the JPL ephemeris DE406, corresponding to the epoch J2000.0. The second model is given by the same numerical ephemeris or simply by approximating the orbits of the other seven planets (excluding the Earth) as circles around the Sun.

Throughout the work, the length unit for computation is the mean distance between the Earth and the Moon (3.847 479 81 × 10^8 km in our work), and the mass unit is the sum of the Earth and the Moon, and the time unit is chosen such that the gravitational constant G = 1. But for all the orbits presented in the synodic frame in the figures, unless specification, the unit is the instantaneous distance between the Earth and the Moon. The numerical integrator used in our work is the usual RKF78 one. The fast Fourier transformation (FFT) analysis is elementary. While some advanced algorithms (Laskar 1999) may be helpful to obtain higher accuracy, we satisfy ourselves with the elementary algorithm because it is simpler and easier to implement and is accurate enough for our study.

2.1 The Sun–Earth–Moon system

Fig. 1 is an illustrative figure showing the geometry between the Earth, the Moon, the Sun, and the small body in the Earth-centred inertial frame. The orbits of the other seven planets (excluding the Earth) are visualized as circles around the Sun. Motions around the TLP are usually visualized in the synodic frame. Denote the transformation matrix from the Earth-centred synodic frame to the Earth-centred inertial frame as C, we have

\[ \mathbf{R} = k \mathbf{C} \mathbf{r}, \quad \mathbf{r} = k^{-1} \mathbf{C}^T \mathbf{R}, \]  

(3)

where k is the instantaneous distance between the Earth and the Moon. Since we are using the mean distance as the length unit, k is around 1. The length unit for the variable r in this case is the instantaneous distance between the Earth and the Moon. Computing the matrix \( \mathbf{C} \) is not a tough task (Gómez et al. 2001a) and the details are not omitted here.

2.2 full force model

The full force model is only used when we test the sustainability of the stable orbits (which are obtained in the above model) with the addition of planetary perturbations. Two full force models are used in our work. The first one is given by the JPL numerical ephemeris DE406. The problem with this model is the length it covers (6000 yr in total, and 1000 yr from the epoch J2000.0). To see the effects of longer integrations, we need a much longer ephemeris. A complete integration of the small body, the Moon, the Sun, and the other seven planets in the Earth-centred inertial frame is not feasible because we have three time-scales: the time-scale in the Earth–Moon system (~1 month), the time-scale of the inner planets (~1 year) and the time-scale of the outer planets (~10 yr). The computation cost is high and the accumulation of truncation errors is unbearable. To simplify our studies, the orbits of the other seven planets are approximated as circles around the Sun. The orbit of the Trojan and the Moon is integrated in the Earth-centred inertial frame, and the orbit of the Earth with respect to the Sun is integrated in the Sun-centred inertial frame.

3 STABLE MOTION DESCRIPTION

In Kolenkiewicz and Carpenter’s work (1968), the stable motion in the planar Sun–Earth–Moo system is numerically searched. In our studies, we did a similar numerical search, but in the spatial Sun–Earth–Moon system. For each TLP, except the two stable regions reported by Kolenkiewicz and Carpenter, we are unable to locate other stable regions. In Kamel’s work (1969), based on the planar Sun–Earth–Moon model, analytical formulae of two special stable orbits for each TLPs were given. In the work of Simó et al. (1995), using the simple model of bicircular problem (BCP), similar stable region (not exactly same due to the incoherence of the BCP) is also found via a numerical approach. In Scheeres’ work (1998), these stable orbits are also found by using the restricted Hill four-body problem. In our numerical computations, besides the above two stable regions, we find that some orbits with very large vertical deviations (with respect to the Moon’s orbital plane) can also stay in the system for quite a long time, which is a phenomenon already reported by Jorba (2000). However, with even longer integration times, it turns out that the eventual fate of these orbits is to escape from the system. These orbits are not interest of this paper.

In this section, we will describe how these stable orbits come into their existence and how we compute them in the spatial Sun–Earth–Moon system. We will also describe the stable region and simulate its destiny in the full force model where planetary perturbations are added.
3.1 Dynamical substitute

Denoting in the synodic frame the deviation of the small body from the TLP as $\rho$, the EOM with respect to the TLP can be written as

$$\ddot{\rho} = F_1(\rho, t) + F_2(t).$$

(4)

The details to deduce these equations are omitted (Hou & Liu 2010). The first term $F_1(\rho, t)$ can be expanded as a literal series of the deviation $\rho$. The second term $F_2(t)$ is irrelevant to $\rho$ and is determined by the motion of the Moon. Under the assumption that the Moon’s motion is quasi-periodic under the Sun’s perturbation, $\rho$ is quasi-periodic, with several basic frequencies. For the Earth–Moon system perturbed by the Sun, there are four basic frequencies and they can be recovered by FFT analysis of the orbits of the Moon under the influence of the Sun. In our work, they are

$$\omega_1 = 0.9915 47 97$$
$$\omega_2 = 0.0748 01 74$$
$$\omega_3 = 0.9251 97 71$$
$$\omega_4 = 1.0040 20 82.$$

(5)

Their values are a little bit different from those in Hou & Liu (2010) because here we are dealing with the Sun–Earth–Moon system but not the full force model given by the numerical ephemera.

Due to the term $F_2(t)$, the geometrical TLPs are no longer equilibrium points. When some conditions are satisfied (Jorba & Villanueva 1997), special quasi-periodic solutions with the same basic frequencies as the perturbing term $F_2(t)$ exist. Expanding equation (4) around these special solutions, we can get rid of the annoying term $F_2(t)$. As a result, these solutions play the role of instantaneous equilibrium points and are called dynamical substitutes or dynamical equivalents (Díez et al. 1991; Chappaz & Howell 2015). They can be considered as forced motions caused by the Sun and the Moon. For the simple BCP, three such solutions are found (Simó et al. 1995). The inner one which is closer to the TLP is unstable. Its companion in the real force model is described by Hou & Liu (2010). The outer two are stable. Their companions in the planar Sun–Earth–Moon system are actually the two stable orbits found by Kolenkiewicz & Carpentier (1968) and Kamel (1969). In this paper, we will focus on the outer two stable orbits.

The same algorithm as the one in Hou & Liu (2010) is used to compute the two dynamical substitutes in the spatial Sun–Earth–Moon system. Generally, stable motions around the dynamical substitutes can be expressed as

$$\begin{align*}
\bar{\xi} &= \sum_{i,j,k,l} \xi_{ijkl}^{(1)} \cos \bar{\theta} + \xi_{ijkl}^{(2)} \sin \bar{\theta} \\
\bar{\eta} &= \sum_{i,j,k,l} \eta_{ijkl}^{(1)} \cos \bar{\theta} + \eta_{ijkl}^{(2)} \sin \bar{\theta} \\
\bar{\zeta} &= \sum_{i,j,k,l} \zeta_{ijkl}^{(1)} \cos \bar{\theta} + \zeta_{ijkl}^{(2)} \sin \bar{\theta},
\end{align*}$$

(6)

where

$$\bar{\theta} = (i\omega_1 + j\omega_2 + k\omega_3 + l\omega_4)t + n_1(v_0t + \theta_0^0) + n_2(v_0t + \theta_0^0) + n_3(v_0t + \theta_0^0).$$

(7)

The frequencies $v_0, v_1, v_2, v_3$ are the basic frequencies of the free motion which will be described in the next subsection. For the dynamical substitute, we have $n_1 = n_2 = n_3 \equiv 0$. That is

$$\begin{align*}
\bar{\xi} &= \sum_{i,j,k,l} \xi_{ijkl}^{(000)} \cos \theta + \xi_{ijkl}^{(100)} \sin \theta \\
\bar{\eta} &= \sum_{i,j,k,l} \eta_{ijkl}^{(000)} \cos \theta + \eta_{ijkl}^{(100)} \sin \theta \\
\bar{\zeta} &= \sum_{i,j,k,l} \zeta_{ijkl}^{(000)} \cos \theta + \zeta_{ijkl}^{(100)} \sin \theta,
\end{align*}$$

(8)

where

$$\theta = \bar{\theta} + \theta_0^0, \quad \theta_1 = v_0t + \theta_0^0, \quad \theta_2 = v_0t + \theta_0^0.$$

The subscripts of the frequencies $v_0, v_1, v_2, v_3$ indicate their origins from the long period, the short period, and the vertical motion around...
the TLP in the circular restricted three-body problem (CRTBP). \(\theta_0^i, \theta_0^v, \theta_0^l\) are initial phase angles which can be arbitrarily chosen. \(\alpha, \beta, \gamma\) are three amplitude parameters which indicate the motion amplitude of each frequency. They can also be arbitrarily chosen but too large values will cause the invalidity of the linear solution due to non-linear terms and resonances discussed below.

3.3 Non-linear terms and resonances

Due to non-linear terms, the basic frequencies \(v_i, v_v, v_l\) vary with amplitude parameters \(\alpha, \beta, \gamma\). We did not make any efforts to obtain higher order analytical solutions for the coordinates. However, we do give the second order corrections to obtain higher order analytical solutions for the coordinates. However, we do give the second order corrections to the basic frequencies because we will need them in following discussions on resonances. Since the two TLPs should have symmetric stable regions under the third body perturbation from the long time perspective (Hou, Scheeres & Liu 2014a), we only concentrate on the TLP L4. We further simplify the studies by only concentrating on one of the two dynamical substitutes. The results presented below are only for the first dynamical substitute around the point L4. The second-order formulae for the frequencies are (Hou, Scheeres & Liu 2014a)

\[
\begin{align*}
v_l &= v_{l00} + v_{l000} \alpha^2 + v_{l002} \beta^2 + v_{l002} \gamma^2 \\
v_v &= v_{v00} + v_{v020} \alpha^2 + v_{v002} \beta^2 + v_{v002} \gamma^2 \\
v_i &= v_{i00} + v_{i020} \alpha^2 + v_{i002} \beta^2 + v_{i002} \gamma^2.
\end{align*}
\] (10)

The coefficients are computed numerically. Using the linear solution in the above subsection, we can compute stable orbits with different amplitude parameters \(\alpha, \beta, \gamma\). We set \(\alpha, \beta, \gamma \in (0, 0.01]\), with a step size of 0.001 in each parameter. Totally 1000 stable orbits are computed. FFT analysis of these orbits leads to a series of \(v_l, v_v, v_i, \alpha, \beta, \gamma\). Numerical fitting of this series using the minimum variance principle gives

\[
\begin{align*}
v_{l000} &= 0.3250710, & v_{v000} &= -0.74980808 \\
v_{l020} &= -0.46199565, & v_{v002} &= -0.63931503 \\
v_{v00} &= 0.94584527, & v_{v20} &= 0.28926803 \\
v_{l020} &= -0.27552939, & v_{v020} &= -0.20950784 \\
v_{l000} &= 0.68573561, & v_{v002} &= 0.60625557 \\
v_{l20} &= -0.03437789, & v_{v020} &= -0.03371370.
\end{align*}
\]

Equation (10) can be used to approximate the frequencies for small amplitude motions. Different from the Sun–Jupiter system (Hou et al. 2014a), the coefficients are very large. This means the basic frequencies \(v_i, v_v, v_l\) change rapidly with larger amplitudes. Various resonances may be easily encountered even when the motion amplitudes are not very large. Different resonances and their overlap with each other are the reason for the instability of the motions (Hou et al. 2014a). This means the stable region may not be large. We will talk about the stable region in the next subsection. Here we describe how these resonances cause the instability of the motion with increasing amplitudes.

To simplify the discussions, we restrict ourselves in the \(x - y\) plane (i.e. \(y \equiv 0\)). A carefully chosen example is given in Figs 3 and 4. The solid curve and the dashed curve in Fig. 4 indicate the two resonances:

\[
\begin{align*}
\Delta \omega_1 &= -7 \omega_1 + 11 \omega_2 + 10 \omega_3 + 2 v_l - 4 v_i \\
\Delta \omega_2 &= -6 \omega_1 + 3 \omega_2 - \omega_3 + 3 v_l + 6 v_i.
\end{align*}
\]
fact, for the Earth–Moon system, the difference between the two frequencies $v_l$ and $v_t$ is not so large and the process in Fig. 3 can also be clearly observed around the short period component.

The resonance structures (i.e. curves generated by various resonances using equation 10) provide passages for motion to diffuse in the phase space (Robutel & Gábedn 2006). Curves of the type $\Delta \omega_1 = 0$ can cause motions to diffuse but still in a confined region in the phase space, but curves of the type $\Delta \omega_2 = 0$ may cause motions to diffuse from inwards to outwards. Since the two curves intersect, it is still possible for motions along the $\Delta \omega_1 = 0$ curve to eventually diffuse outwards along the $\Delta \omega_2 = 0$ curve. For motions of small amplitudes, the diffusion speed may be extremely slow due to the fact that the resonance strength is so small. The chaotic motion may be confined to the proximity of the regular orbits which may be densely distributed in the phase space. But with the motion amplitudes increasing, the strength of the resonances increases. The number of the regular orbits surviving these resonances becomes much less. This makes the chaotic diffusion easier and faster. Besides, with the motion amplitudes increasing, it's easier for different resonances to overlap with each other. This makes the diffusion even easier. A good intuitive picture is provided in Hou et al. (2014a, fig. 12).

One remark should be made. Due to the large values of $v^{002}_l, v^{002}_t, v^{002}_v$, the two planar frequencies $v_l, v_t$ are also greatly influenced by the out-of-plane amplitude $\gamma$. Besides, when three-dimensional motions are considered, the frequency $v_t$ will involve in and more resonance terms will appear. This makes the resonance structures in the phase space much more complicated.

### 3.4 Stable region

The above subsection describes the resonance mechanism that causes the instability of the motion. A thorough survey of these resonance structures and associated resonance widths may help identify the stable region around the dynamical substitute. However, these structures are very complicate, especially when the vertical component is involved in. We take an easier numerical way. Different orbits with different amplitude parameters $\alpha, \beta, \gamma$ are integrated within a fixed time to observe whether they are unstable or not. The criterion of instability is the intersection of the integrated orbit with the $x - z$ plane. The integration time is one million years. In the authors' opinion, this time is long enough compared with the characteristic time in the Earth–Moon system which is about 1 month. To accelerate the computation, the multistep integrator (Quinlan & Tremaine 1990) is used. We did not try to study the whole spatial stable region but two subcases: the out-of-plane stable region ($\alpha, \beta \equiv 0, \gamma \neq 0$) and the in-plane stable region ($\alpha, \beta \neq 0, \gamma \equiv 0$).

It is easier for the out-of-plane case. We simply set $\alpha, \beta \equiv 0$ and vary $\gamma$ in equation (9) to search for the maximum allowable value for the stable region. Our study shows that $\gamma_{\text{max}} \sim 0.117$. Fig. 5 shows an orbit with $\gamma = 0.116$. For the reason of clarity, only the first 100 years are presented. The parameter $\gamma$ can be taken as a parameter for the orbit inclination of the small body with respect to the Moon’s orbit plane.

For the in-plane case, it’s a little bit complicate. Now, we have two parameters. For each fixed $\beta$, there is a maximum value of $\alpha$ for stable motions (within the integration time). When a different value for the parameter $\beta$ is chosen, the new maximum value of $\alpha$ changes accordingly. Fig. 6 shows the in-plane stable region. A step size of 0.02 in the parameter $\beta$ is taken. The filled dots are for an integration time of $10^6$ years and the unfilled circles are for an integration time of $10^8$ years. Even with integration times of 10 times difference, the stable region remains nearly unchanged.

One remark should be made. Due to strong non-linear effects, the linear approximation equation (9) is not good even for moderate $\alpha, \beta$ or $\gamma$. Meanwhile, constructing non-linear analytical solutions is difficult. To solve this problem, the initial conditions are still given by equation (9), but are numerically refined by the multiple shooting method (Gómez et al. 2001a) before integration. In the authors’ opinion, by numerically forcing the orbit to be quasi-periodic (at least within some time) may partially compensate the non-linear terms that are not included in the linear approximation. A number of 257 nodal points with an interval of 1 d is used in the multiple shooting algorithm.

The above stable regions are only approximate due to following reasons.

(i) Only linear formulae are used to approximate the motions nearby the dynamical substitute, although they are refined by the multiple shooting algorithm. For small values of $\alpha, \beta, \gamma$, the linear approximation is good. But for larger values, two problems appear due to non-linear terms. The first one is that there is a deviation between the true amplitude parameters $\alpha, \beta, \gamma$ and the ones in the linear formulae. The second one is that different types of motion cannot be cleanly separated. For example, in the linear formulae, we set $\alpha = \beta = 0$ and vary $\gamma$ to get initial conditions for the ‘vertical’ orbits, but the truly integrated orbits may have $\alpha, \beta \sim \gamma^2$ due to non-linear effects (Hou & Liu 2010).

(ii) Only the in-plane stable region and the out-of-plane stable region are obtained. The stable region in the three-dimensional space is not described. More accurately speaking, we even did not give the stable regions but only the boundaries. It is still possible for orbits within this boundary to become unstable due to the resonances described in the above subsection.
Figure 7. The span angle in space of the in-plane stable region (left-hand) and the out-of-plane stable region (right-hand) with respect to the dynamical substitute.

(iii) The stable boundaries described are only the results within a fixed integration time, although a longer integration time may not modify the results too much according to Fig. 6.

Nevertheless, in the authors’ opinion, the results obtained above are still good approximations for the overall description of the in-plane and out-of-plane stable regions. If there are objects there, they should be confined within this region if no perturbations other than gravitational forces contribute.

Stable region described in Figs 5 and 6 are not intuitive for observations. In observations (see Section 5), we care about how large in space the stable region can cover. Fig. 7 shows the span angle of the planar stable region (left-hand) and the vertical stable region (right-hand) with respect to the dynamical substitute. Totally 27 orbits at boundary of the planar region (see Fig. 6) and one orbit at the boundary of the vertical region (see Fig. 5) are integrated for 10 000 d. Every one day, a sample point is taken. Judging from this figure, we have a maximum span angle (with respect to the dynamical substitute) of about 8° for the planar stable region and of about 7° for the vertical stable region. If we want to search temporarily captured objects (see Section 5), a larger span angle should be covered by telescopes.

3.5 Full force model

All the results described above are for the Sun–Earth–Moon system. In the real Solar system, the other seven major planets also contribute. The contributions are twofolds: the direct gravitational perturbation on the small body and the indirect perturbation induced by modifying the Earth’s orbit with respect to the Sun and the Moon’s orbit with respect to the Earth.

One approach to study the effects of the planetary perturbations is to use the numerical ephemeris. The dynamical substitute is integrated in the full force model to see whether it is unstable or at least shows a sign of chaos. Taking one of the dynamical substitutes around the point L4 as an example, Fig. 8 shows local details around the short-period component and the long-period component. The continuous frequency map is clean, at least for an integration time of 1000 years starting from the epoch J2000.0. There are two reasons for the non-zero short-period component (and also the long-period component) in Fig. 8. First, the initial conditions are given by FFT approximations of the dynamical substitute. The FFT series is truncated at a finite order (33 terms for the x, y components and 11 terms for the z component), so the accuracy is limited. Secondly, these analytical formulae are for the Sun–Earth–Moon system while the integration model is the full force model. Even though, the amplitudes of the free motions (the short-period component and the long-period component) are small, which means the analytical formulae are good approximations of the dynamical substitutes in the

Figure 8. Local details around the long-period component (left-hand) and the short-period component (right-hand) in the frequency map for a ‘dynamical substitute’ integrated in the full force model using the numerical ephemeris.

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Figure 9. Continuous Fourier transform map for the orbit integrated in the intermediate model (solid lines) and the orbit integrated in the Sun–Earth–Moon system (dotted lines).

full force model. To obtain more accurate approximations of the dynamical substitutes in the full force model, the same iteration method as described in Section 3.1 can be used. The results in the appendix are given in this way.

The problem with the above approach is that the length of the numerical ephemeris is too short. A complete integration model consisting of the small body, the Moon, the Sun, and the eight planets takes a very long integration time and may still be inaccurate due to numerical issues and other unconsidered perturbations in practice. The main goal of this study is to observe qualitatively the effects of the planets on the Sun–Earth–Moon–small body system but not the accurate orbits of the planets, so we take an intermediate model. The other seven planets are assumed moving on coplanar circular orbits around the Sun in the ecliptic plane. The Earth’s orbit around the Sun and the Moon’s orbit around the Earth are perturbed by them. The small body’s motion is simultaneously influenced by the Sun, the Earth, the Moon, and the other seven planets. The initial conditions of the Sun and the Moon with respect to the Earth are given by the numerical ephemeris. The semimajor axes and the initial phases of the other seven planets are taken from table A.2 in the appendix of Murray & Dermott’s book (1999), corresponding to the epoch J2000.0. Also taking the first dynamical substitute around the point L4 as an example and integrating the orbit for 10^7 years, no sign of instability can be observed. Fig. 9 shows the details of the frequency map (solid curves) for a sample data of 2^22 points with an interval of 5 d. The map is generated by the continuous Fourier transform. The frequency map is clean. No obvious chaos can be observed. For comparison, the results for the orbit integrated in the Sun–Earth–Moon system with the same initial condition are shown in dotted curves. Except a small shift of the basic frequencies, the two frequency maps do not show any obvious difference.

As to the problem of stable region in the full force model, we did not make any efforts to compute it due to the heavy computation burden. Our opinion is that the stable region may change a little bit
at the boundary due to the addition of extra perturbation frequencies and other possible resonances, but generally remains unaltered.

4 TEMPORARY CAPTURE

Studies above show that the stable region is not large, and thus the probability to find natural objects in this stable region is low. However, for orbits out of the stable region but close to the stability boundary, objects may be temporarily trapped around this region for a long time. Fig. 10 shows two examples. The orbits are integrated backwards in the Sun–Earth–Moon system. The upper frames are for the Earth–Moon synodic frame (centred at the Earth) and the lower frames are for the Sun–Earth synodic frame (centred at the Sun). For clarity, integration time of the upper-left frame is $-6 \text{ yr}$, and integration time of the upper-right frame is $-3.5 \text{ yr}$. Length unit in the above two frames is the instantaneous distance between the Earth and the Moon, and length unit in the lower two frames is the instantaneous distance between the Sun and the Earth. The arrows in the lower frames indicate the motion direction.

The above orbits are just two examples out of a number of numerical tests. The initial conditions of these tested orbits are given by formulae equations (8)–(10), with parameters $\alpha$, $\beta$, $\gamma$ and $\theta^0$, $\phi^0$, $\psi^0$ to make sure they become unstable within an integration time ($-100 \text{ years}$ in this study). The initial conditions for the Earth, the Moon, and the Sun are given by the JPL ephemeris DE 406, corresponding to the epoch J2000.0. Nearly all the backwards integrated orbits eventually go out of the Earth–Moon system, ending up with orbits with semimajor axes close to that of the Earth. The trajectories in the Sun–Earth synodic frame (centred at the Sun) are like the ones in Fig. 11 (corresponding to Fig. 10).

Figs 10–11 mean that near Earth objects (NEO) may find their way to the proximity of the TLPs in the Earth–Moon system via these trajectories. Usually they should first be captured by the Earth (via the L1 or L2 points in the Sun–Earth system, see Fig. 11), and then enters the Earth–Moon system (see Fig. 10), but it is not always the case. If the energy is large enough, these objects can directly enter the Earth–Moon system without passing through the proximity of the collinear libration points of the Sun–Earth system. One example is shown in Fig. 12.

Several remarks should be made here. The first remark is that the temporary capture orbits are unstable. The instability cannot be strong because otherwise the object will leave this region very quickly and cannot be called a temporary capture. The unstable orbits around the stable dynamical substitutes (such as the ones in Figs 10 and 12) are not the only choice for a temporary capture orbit. In previous works (Gómez et al. 2001b; Hou & Liu 2010), the instability of the dynamical substitute close to the TLP in the Earth–Moon system is also shown to be weak. So theoretically speaking, NEOs entering the Earth–Moon system can also be temporarily captured around this region (i.e. a region closer to the TLP) for some time.

The second remark is that objects temporarily captured in the Earth–Moon system via above trajectories are different from the quasi-satellites of the Earth (Mikkola et al. 2006; Kortenkamp 2013). Quasi-satellites are actually objects on a heliocentric orbit with semimajor axes equaling that of the Earth. They do not change much if the gravities of the Earth and the Moon are absent. But objects temporarily trapped on the trajectories studied in this section closely depend on the gravities of the Earth and the Moon. They are Earth’s temporary satellites trapped in the 1:1 resonance with the Moon. They can be classified as one kind of temporarily captured orbiters (TCO) by the Earth (Granvik, Vaubaillon & Jedicke 2012; Bolin et al. 2014).

The third remark is that these temporary capture trajectories may have potential applications in space missions. Currently, the asteroid redirect mission (ARM) aims at retrieving a small near Earth asteroid (or boulders on a bigger asteroid) to the Earth–Moon system (Brophy, Culick & Friedman 2012; Brophy & Muirhead 2013; Strange et al. 2013; Yáñez, Sanchez & McInnes 2013). One possible candidate orbit for holding the retrieved asteroid is a stable lunar orbit. In our opinion, the lunar orbit can be replaced by the stable orbits studied in this paper. Via the temporary capture trajectories, the orbit insertion maneuver to these stable orbits usually requires less fuel than the one to the lunar orbit. In Section 6, further discussions on some possible applications of these libration points in space missions will be made.
5 OBSERVATIONS

If there are (permanent or temporarily captured) objects around these dynamical substitutes, they should generally be of small size, because it is unlikely that large objects remain mysterious to us nowadays. They may accumulate together, behaving as a ‘cloud’. Generally, if this is the case, our study above may not be accurate enough because the internal forces between these close-in bodies may not be negligible. Also, the solar radiation pressure should be considered for small particles or dusts.

Since these suspicious objects should be of small sizes, their apparent magnitude in space is low. As a result, they cannot be easily observed. Fig. 13 is a map showing the relative geometry between the Earth, the Moon, the Sun, and the libration points. The map is produced by looking from the north pole towards the ecliptic plane.

(i) During the new moon, it is better to observe the L4 point because the L5 point may submerge in the zodiacal light. Similarly, during the waning moon, it is better to observe the L5 point because the L4 point may submerge in the zodiacal light.

(ii) For the full moon, it is also possible that the L4 or L5 points are affected by the strong moonlight. Nevertheless, the influence may not be severe. Judging from Fig. 2, the small body cannot approach the Moon very closely.

(iii) From the new moon to the full moon, the L4 point may submerge in the counterglow. Similarly, from the full moon to the waning moon, the L5 point may submerge in the counterglow.

If objects exist, they may not exactly follow the dynamical substitutes, but generally are supposed to be confined in the vicinity of the dynamical substitute. It is best to search these areas by telescopes with a large field, and by putting the dynamical substitute at the centre of the field. In the appendix, the initial conditions of the four dynamical substitutes in the J2000.0 Earth-centred celestial frame are given. Using the initial conditions, it is possible to calculate the available time for observations. The main point is that the observation area has a relative large angle with respect to the Sun (to avoid the zodiacal light), and with respect to the antisolar point (to avoid the counterglow). Besides, they cannot be in the shadow of the Earth or the Moon.

An observation strategy is described here. Suppose 25 fields in the sky (it is just an example, any value can be taken depending on the ability of the telescope) are taken in one observation, with an arrange order as Fig. 14. Suppose the field of the telescope is \( a \times b \), then the total field is \( 5a \times 5b \) for each observation. For each field in Fig. 14, the observation time is \( T \), starting from the epoch J2000.0.

Then one observation totally lasts \( 25T \). As a result, for each field, the observation time slot is

\[
[t_1, t_1 + T], \quad [t_2, t_2 + T], \ldots, [t_{25}, t_{25} + T]
\]

Since some time is needed to adjust the telescope from one field to another, usually \( t_i \geq (t_{i-1} + T) \).

For the dynamical substitute to be at the centre of the total observation area, the field ‘1’ should have the dynamical substitute at its centre. To do so, the following strategy is taken. The longitude and latitude of the dynamical substitute in the observatory-centred equatorial frame are functions of the time \( t \). Denote them as \( \alpha(t) \) and \( \delta(t) \). For each field ‘\( i \)’, we can calculate the positions of the dynamical substitute in the sky at the starting epoch as \( \alpha(t_i), \delta(t_i) \).

As a result, the position of the centre of the filed ‘1’ in the sky at the epoch \( t_i \) is

\[
\alpha_i = \alpha(t_i), \quad \delta_i = \delta(t_i)
\]

However, that is not the direction of the telescope. The direction angle of the telescope should be

\[
\alpha_i = \alpha_i + \Delta \alpha_i = \alpha(t_i) + \Delta \alpha_i, \quad \delta_i = \delta_i + \Delta \delta_i = \delta(t_i) + \Delta \delta_i,
\]

where \( \Delta \alpha_i, \Delta \delta_i \) is the relative deviation of the field ‘\( i \)’ from the filed ‘1’ which can be easily obtained from Fig. 14. So the direction angle of each filed is

\[
(\alpha_1, \delta_1) = (\alpha(t_1), \delta(t_1)), \quad \alpha_2 = (\alpha(t_2) + a, \delta(t_2))
\]

\[
(\alpha_3, \delta_3) = (\alpha(t_3) + a, \delta(t_3) - b) \quad (\alpha_4, \delta_4) = (\alpha(t_4) + a, \delta(t_4) - b) \quad \ldots \quad (\alpha_{25}, \delta_{25}) = (\alpha(t_{25}) + 2a, \delta(t_{25}) + 2b)
\]

One remark is that the longitudes and latitudes should correspond to the coordinate centred at the observatory.

For each filed ‘\( i \)’, the direction of the telescope is held constant \( (\alpha_i, \beta_i) \) during the time slot \( [t_i, t_i + T] \), i.e. the telescope observes the same area in the sky. During the time slot, two or three films can be taken and compared with each other to identify objects.

In the Earth-centred coordinate, Fig. 15 shows the longitude and the latitude variation rate of one dynamical substitute around the point L5, starting from the epoch J2000.0. The objects, if exist, are believed to have similar variation rate because they should be at the proximity of the dynamical substitute. The variation rate of the
longitudinal is larger than that of the latitude. Taking the minimum value (about 0.0001 degrees per second) in the left-hand frame as an example, if the observation time of each filed is $T = 5$ min, the total drift of the object in the films is about 0.03, which should be observable if their apparent magnitude is large enough to be filmed in such a time length.

In fact, using the planar analytic approximations, Freitas & Valdes (1980) already made some observations around these dynamical objects, with no positive results. Since the analytic approximations they used are not accurate enough and neglect the out-of-plane motion, we think it is worth to carry on the observations again with the results presented in this paper and more powerful telescopes (for example, a space-based telescope).

6 SPACE APPLICATIONS

Even if there are no objects around these points in the Earth–Moon system, they are of interest in space missions. The stable orbits discussed in this work are ideal places for space colonies because no orbit control is needed. These stable orbits can also be used as nominal orbits for space VLBI (Very Long Baseline Interferometry) stations. Fig. 16 shows such a VLBI constellation. For practical uses, not all the stations are necessary. For the same TLP, as mentioned in Section 3.1, there are several choices for the nominal orbits (Hou, Tang & Liu 2015). Here we list some. To save space, only one of them is provided with figures.

(1) The second term $F_2(t)$ in equation (4) is irrelevant to $\rho$. Studies show that the TLPs are stable if $F_2(t)$ is removed. In practice, if we use low thrust to cancel this term, then the spacecraft can stay around the TLP. The amplitude of the nominal orbit depends on the initial conditions. The advantage is that the low thrust is only a function of the time and is irrelevant to the specified orbit.

(2) The inner dynamical substitute described in Hou & Liu (2010) is unstable. Spacecraft can also stay around this dynamical substitute. Generally, orbit control is necessary in this case. However, the instability is weak (Hou & Liu 2010). The frequency to execute the control is low (Hou, Tang & Liu 2010).

(3) The outer two dynamical substitutes described in this paper are stable. They can be used as nominal orbits for spacecraft. Generally, no orbit control is required.

(4) Some orbits with a very large out-of-plane deviation (with respect to the Moon’s orbital plane) can stay around the TLPs for quite a long time (1000 yr or even longer, see Jorba 2000). According to our studies (see Section 3), these orbits are actually unstable, but the stability is good enough for practical uses (i.e. no orbit control is needed).

(5) Some orbits such as the one in the left-hand frame of Fig. 17 can periodically get close to and far away from the outer two dynamical substitutes. This kind orbit can visit a very large region around the TLP and is useful when exploring the space situations there. The orbit in Fig. 17 is unstable, but it can last more than 100 years around the TLP (see the right frame of Fig. 17, for the $x$ component). The stability property is also good enough for practical uses.

As to transfer trajectories to the proximity of these points, there are some previous work (Salazar, Macau & Winter 2014; Zhang & Hou 2015). Although nearly all the work is restricted to be in the CRTBP model of the Earth–Moon system or the BCP model of the Sun–Earth–Moon system, they are good approximations of the real system. These studies show that with the aid of lunar gravity or solar gravity, low energy transfer orbits to these points are possible. In the real Sun–Earth–Moon system, this conclusion still holds. The details are omitted here.

7 CONCLUSIONS

In the paper, two special stable quasi-periodic orbits called dynamical substitutes around each TLP in the Earth–Moon system are
computed in the spatial Sun–Earth–Moon system. Stable regions and temporary capture phenomenon around them are studied. One observations strategy is given, followed by a brief discussion on their potential applications in future space missions.

If there are objects around these dynamical substitutes, observations are necessary. Are these objects of primordial origin (Cuk & Gladman 2009), or being captured after the formation of the Earth–Moon system? Is it possible for some artificial objects such as space debris or dust particles to evolve to these places (Struck 2007)? Even if there are no objects there, observations are still necessary for future space applications because we have to be very sure of the space situations before we send probes to these places. The observations should not be confined to the stable regions reported in this paper, because the current study neglects the solar radiation pressure and the internal forces between small bodies which may be important when the bodies are very small. But still, the dynamical substitutes reported in this paper are good places to start the observations.

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APPENDIX

We provide positions (km) and velocities (km s$^{-1}$) of the four dynamical substitutes in the Earth-centred J2000.0 equatorial coordinate, corresponding to the epoch J2000.0.

(i) The first dynamical substitute around the point L4:

\begin{align*}
x &= 100 036.439 532 585, & y &= -420 392.379 622 313 \\
z &= -170 330.997 700 722
\end{align*}

\begin{align*}
x &= 0.828 546 920 510 318, & y &= 0.171 131 013 465 078 \\
z &= 0.020 930 811 587 838 46
\end{align*}

(ii) The second dynamical substitute around the point L4:

\begin{align*}
x &= 94 334.967 889 461, & y &= -298 114.348 943 529 \\
z &= -121 989.314 154 805
\end{align*}

\begin{align*}
x &= 1.117 119 425 861 61, & y &= 0.293 433 955 606 099 \\
z &= 0.051 975 997 400 319 28
\end{align*}

(iii) The first dynamical substitute around the point L5:

\begin{align*}
x &= 368 851.040 140 635, & y &= 416 634.659 832 89 \\
z &= 22 682.917 100 298 8
\end{align*}

\begin{align*}
x &= -0.221 086 238 094 874, & y &= -0.962 988 093 016 523, \\
z &= -0.366 698 119 231 776
\end{align*}

(iv) The second dynamical substitute around the point L5:

\begin{align*}
x &= 340 993.663 821 844, & y &= 245 461.452 560 229 \\
z &= 116 072.550 646 332
\end{align*}

\begin{align*}
x &= -0.056 045 259 675 314, & y &= -0.699 126 727 992 046, \\
z &= -0.245 283 859 415 135
\end{align*}

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